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THE TRANSMISSION OF A FLEXURAL-GRAVITATIONAL WAVE THROUGH A RIGID END-STOP IN A FLOATING PLATE†

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An elastic plate is located on the surface of a liquid, in continuous contact with it and rigidly clamped in a support along a certain straight line. The orthogonal incidence of a small amplitude flexural-gravitational wave on the support is considered. Exact expressions are obtained for the wave field in the fluid and the flexural field in the plate. The transmission coefficient of the incident flexural-gravitational wave through the support and its reflection coefficient from it are determined. The forces which arise in the support are found. The investigation is carried out for liquids of finite and infinite depth. The effect of the depth of the liquid on the wave processes is indicated. The liquid is assumed to be inviscid and its friction on the bottom and the lower surface of the plate in the neighbourhood of the support is therefore ignored. © 2002 Elsevier Science Ltd. All rights reserved.

The problem of the interaction of gravitational waves with a floating plate has been extensively investigated as applied to wave processes occurring in a sea covered with ice. For example, flexural-gravitational waves in a liquid with a homogeneous ice sheet have been studied in [1], the problem of the normal incidence of surface waves on the edge of solid ice has been solved in the case of a tank of finite depth [2] and the diffraction of plane waves at one or several cracks in an ice sheet has been considered in [3]. The interest which has recently been shown abroad in the study of the hydro-elastic behaviour of floating plates is primarily due to the planning and construction of water airfield's in a number of foreign states such as the USA and Japan, for example. Hence, problems concerned with the normal and oblique incidence of a gravitational wave on the edge of a floating plate have been considered [4–8]. It is of interest to investigate the effect which different structural components of water airfields and, in particular, the methods used to fix them, have on the wave processes.

A model of a floating platform with the simplest form of fixing, a rigid end-stop along a certain line, is considered. Here, free movement of the liquid particles across the support is assumed to be possible. A flexural-gravitational wave, which is incident orthogonally on the straight line along which the plate is fixed, is the source of the wave field in the water and the flexural field in the plate. If the plate is isolated, the flexural wave is completely reflected from the rigid end-stop. In the case under consideration, a plate and the mass of water under it participate in a common motion. The existence of two channels for energy transmission leads to the partial transmission of the flexural gravitational wave. This paper is concerned with the study of this transmission.

1. FORMULATION OF THE PROBLEM

Consider a plate located on the surface of a liquid and fastened in a support. The Oxy plane coincides with the lower surface of the plate in its unperturbed state. The z axis is directed vertically upwards. The origin of the system of coordinates is located on the straight line of the fixing of the plate. A flexural-gravitational wave propagates along the x axis. With this choice of the system of coordinates, the field is independent of the y coordinate and, consequently, we can consider a two-dimensional problem. The scheme of the model is shown in Fig. 1. Here, h is the depth to which the plate sinks (is immersed) and H is the distance between its lower surface and the bottom.

We shall confine ourselves to considering harmonic processes. The factor $e^{-i\omega t}$ (ω is the angular frequency), which specifies the time-dependence of the processes, is omitted everywhere.

The liquid is assumed to be ideal and incompressible and the wave process in it is assumed to be potential. The required potential in the whole of the volume of the liquid must satisfy the Laplace condition

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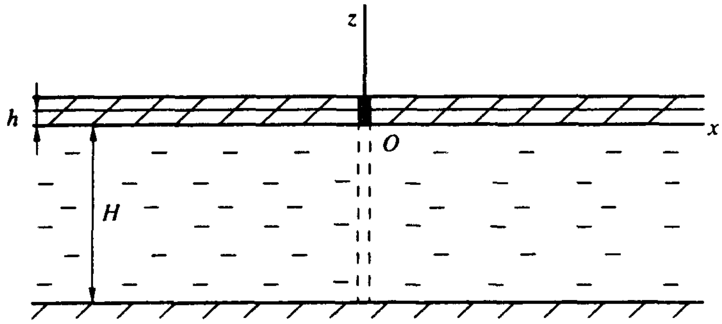

Fig. 1

$$\frac{\partial^2\Phi(x,z)}{\partial x^2}+\frac{\partial^2\Phi(x,z)}{\partial z^2}=0, \qquad -\infty<x<+\infty, \qquad -H\leqslant z\leqslant 0 \tag{1.1}$$

The equation for the equilibrium of the plate under the action of inertial forces, internal elastic forces and the hydrodynamic pressure $P(x, z)$ on the lower surface of the plate is written as

$$D\zeta''''(x)-\rho h\omega^2\zeta(x)-P(x,0)=0 \tag{1.2}$$

where D is the cylindrical stiffness of the plate, ζ is the vertical displacement of the plate, ρ is the density of the liquid and ρh is the surface density of the plate, which is defined on the basis of Archimedes law. Equation (1.2) holds everywhere with the exception of the line along which the plate is fixed ($x = 0$), in which the conditions for the stiffness of the end-stop

$$\zeta(0)=0, \qquad \zeta'(0)=0 \tag{1.3}$$

must be satisfied.

Consequently, there is a function on the left-hand side of Eq. (1.2), the carrier of which is concentrated at the single point $x = 0$. This function is a linear combination of a δ-function and its derivatives. In fact

$$D\zeta''''(x)-\rho h\omega^2\zeta(x)-P(x,0)=A\delta(x)+B\delta'(x) \tag{1.4}$$

Here, we shall confine ourselves to the two terms of the linear combination containing the zeroth and first derivatives of the δ-function. The higher derivatives correspond to the breakdown of the integrity of the plate. The physical meaning of the coefficients A and B will be explained below (Section 5).

In the case of a wave of small amplitude, the hydrodynamic pressure on the lower surface of the plate is given by the formula

$$P(x,0)=-\rho g\zeta(x)+i\omega\rho\Phi(x,0) \tag{1.5}$$

The normal component of the velocity of the liquid on its surface must be identical to the velocity of displacement of the surface of the plate, that is

$$\frac{\partial\Phi(x,0)}{\partial z}=-i\omega\zeta(x) \tag{1.6}$$

Substituting expression (1.5) into the equation for the equilibrium of the plate (1.4) and taking account of the linearized kinematic condition (1.6), we obtain the boundary condition on the lower surface of the plate

$$\left.\left(D\frac{\partial^5\Phi}{\partial x^4\partial z}-\rho h\omega^2\frac{\partial\Phi}{\partial z}+\rho g\frac{\partial\Phi}{\partial z}-\rho\omega^2\Phi\right)\right|_{z=0}=-i\omega(A\delta(x)+B\delta'(x)) \tag{1.7}$$

In the case of a liquid of finite depth, the bottom will be assumed to be rigid

$$\frac{\partial\Phi(x,z)}{\partial z}=0, \qquad z=-H \tag{1.8}$$

The condition at infinity, according to which the required field, after subtracting the incident wave, must consist of waves departing to infinity and decaying at infinity, completes the formulation of the problem.

2. CONSTRUCTION OF THE SOLUTION. THE CASE OF A LIQUID OF FINITE DEPTH

We will seek the velocity potential in the form of the sum

$$\Phi(x,z)=\Phi_0(x,z)+\Psi(x,z) \tag{2.1}$$

where Φ_0 is a surface wave which is incident on the support and ψ is the set of waves scattered by the support. Each of the terms on the right-hand side of Eq. (2.1) must satisfy the system of equations (1.1), (1.7) and (1.8).

The expression for the incident wave is determined using the formula

$$\Phi_0(x,z)=C\,\mathrm{ch}[\lambda_0(H+z)]e^{i\lambda_0 x} \tag{2.2}$$

where C is its amplitude and λ_0 is the wave number ($\lambda_0>0$).

In order to find the scattered wave, we successively apply a direct and an inverse Fourier transformation with respect to the x coordinate to system of equations (1.1), (1.7) and (1.8), which are written for the function ψ. We obtain the following integral representation

$$\Psi(x,z)=-\frac{i\omega}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{ch}[\lambda(H+z)](A+iB\lambda)e^{i\lambda x}}{\Delta(\lambda)}d\lambda \tag{2.3}$$

$$\Delta(\lambda)\equiv D\lambda^5\,\mathrm{sh}(\lambda H)-\rho h\omega^2\lambda\,\mathrm{sh}(\lambda H)+\rho g\lambda\,\mathrm{sh}(\lambda H)-\rho\omega^2\,\mathrm{ch}(\lambda H)$$

The equality of the denominator of the integrand in (2.3) to zero represents the dispersion equation which has two real roots ($\pm\lambda_0$), a denumerable set of imaginary roots and two pairs of complex-conjugate roots. The real roots $+\lambda_0$ and $-\lambda_0$ correspond to the wave numbers of progressive non-decaying waves. The denumerable set of imaginary roots characterizes waves which decay exponentially with distance from the support. The two pairs of complex conjugate roots define progressive decaying waves due to the flexural stiffness of the plate.

The integration in (2.3) is carried out along the real axis passing around the positive root of the denominator of the integrand ($+\lambda_0$) from below and the negative root ($-\lambda_0$) from above (the principle of limiting absorption).

Using the theorem of residues and taking account of the evenness of the function $\Delta(\lambda)$, it is possible to obtain an expression for $\psi(x,z)$ in the form of a sum. As a result, we have the following representation of the wave field in the liquid (everywhere henceforth summation is carried out from $n=0$ to $n=\infty$)

$$\Phi(x,z)=C\,\mathrm{ch}[\lambda_0(H+z)]e^{i\lambda_0 x}-i\omega\sum\frac{\mathrm{ch}[\lambda_n(z+H)](Ai-B\lambda_n\,\mathrm{sign}\,x)e^{i\lambda_n|x|}}{\Delta'(\lambda_n)} \tag{2.4}$$

where λ_n ($n>0$) are the roots of the function $\Delta(\lambda)$, which are located in the upper complex half-plane.

Using the kinematic condition (1.6), we obtain an expression for the flexural field in the plate

$$\zeta(x)=\frac{i}{\omega}C\lambda_0\,\mathrm{sh}(\lambda_0 H)e^{i\lambda_0 x}+\sum\frac{\mathrm{sh}(\lambda_n H)(Ai\lambda_n-B\lambda_n^2\,\mathrm{sign}\,x)e^{i\lambda_n|x|}}{\Delta'(\lambda_n)} \tag{2.5}$$

Finally, the unknown coefficients A and B are determined from the conditions for the rigidity of the end-stop, which the vertical displacements of the plate $\zeta(x)$ must satisfy

$$A=-\Lambda_1, \quad B=i\Lambda_2 \tag{2.6}$$

$$\Lambda_k = \frac{1}{\omega} C\lambda_0^k \operatorname{sh}(\lambda_0 H) \left[\sum \frac{\lambda_n^{2k-1} \operatorname{sh}(\lambda_n H)}{\Delta'(\lambda_n)} \right]^{-1}, \quad k = 1,2 \tag{2.7}$$

3. CONSTRUCTION OF THE SOLUTION. THE CASE OF AN INFINITELY DEEP LIQUID

The solution is constructed using a technique which is similar to that presented above. Instead of the impermeability condition on the bottom (1.8), a condition is used here which requires that, at a great depth, the liquid should be in a state of rest: $\Phi(x, z) \to 0, z \to -\infty$. Hence, for the incident wave, we have

$$\Phi_0(x,z) = Ce^{i\lambda_0 x} e^{\lambda_0 z} \tag{3.1}$$

where λ_0 is the positive root of the dispersion equation

$$\Delta(\lambda) \equiv D\lambda^5 - \rho h\omega^2\lambda + \rho g\lambda - \rho\omega^2 = 0 \tag{3.2}$$

The scattered wave is determined from the formula

$$\Psi(x,z) = -\frac{i\omega}{2\pi} \int_0^\infty \frac{(A - iB\lambda)e^{-i\lambda x} + (A + iB\lambda)e^{i\lambda x}}{\Delta(\lambda)} e^{\lambda z} d\lambda \tag{3.3}$$

As in the previous case, the positive root of the numerator of the integrand is passed from below.

On satisfying the kinematic condition (1.6), we have an expression for the displacement field in the plate

$$\zeta(x) = \frac{i}{\omega} C\lambda_0 e^{i\lambda_0 x} + \frac{1}{2\pi} \int_0^\infty \frac{(A - iB\lambda)e^{-i\lambda x} + (A + iB\lambda)e^{i\lambda x}}{\Delta(\lambda)} \lambda d\lambda \tag{3.4}$$

The coefficients A and B are determined using formulae (2.6) in which

$$\Lambda_k = \frac{\pi i}{\omega} C\lambda_0^k \left[\int_0^\infty \frac{\lambda^{2k-1} d\lambda}{\Delta(\lambda)} \right]^{-1}, \quad k = 1,2 \tag{3.5}$$

4. TRANSMISSION OF THE INCIDENT WAVE

The amplitude transmission and reflection coefficients of the incident wave, C_T and C_R respectively, are calculated using the formulae

$$C_T = 1 - \frac{i\omega}{C} \frac{Ai - B\lambda_0}{\Delta'(\lambda_0)}, \quad C_R = -\frac{i\omega}{C} \frac{Ai + B\lambda_0}{\Delta'(\lambda_0)} \tag{4.1}$$

For a quantitative estimate of the effect of such factors as the period of the wave and the depth of the reservoir on the transmission of the incident wave, a series of numerical experiments was carried out for a plate with stiffness $D = 3.33 \times 10^{10}$ N m, a liquid (water) with density $\rho = 1025$ kg/m^3 and an immersion of the plate $h = 3$ m.

The dependence of the energy transmission coefficient T of the incident wave and the relative flow of energy in the water $E = \Pi_w/(\Pi_w + \Pi_p)$ (Π_w and Π_p are the energy flows in the water and in the plate) on the period of the incident wave τ for different depths of the water reservoir M ($M = H + h$) is shown in Fig. 2. It can be seen that the transmission coefficient becomes larger as the period of the wave increases. This can be explained as follows. The mechanical resistance of the plate decreases as the period increases. The relative fraction of the energy flow which is carried by the flexural wave in the plate decreases. The water begins to play a fundamental role in the transfer of energy, which leads to an increase in the transmission coefficient since, according to our assumption, the support does not

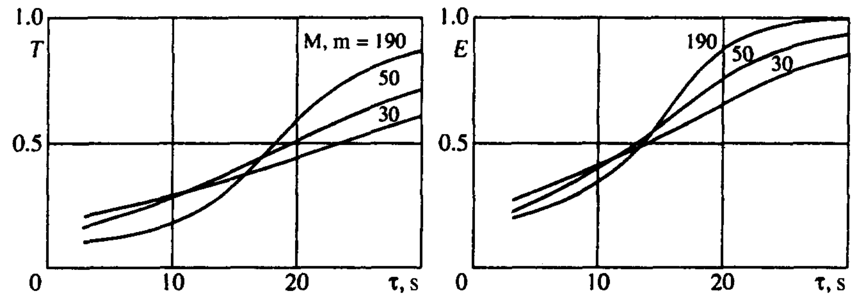

Fig. 2

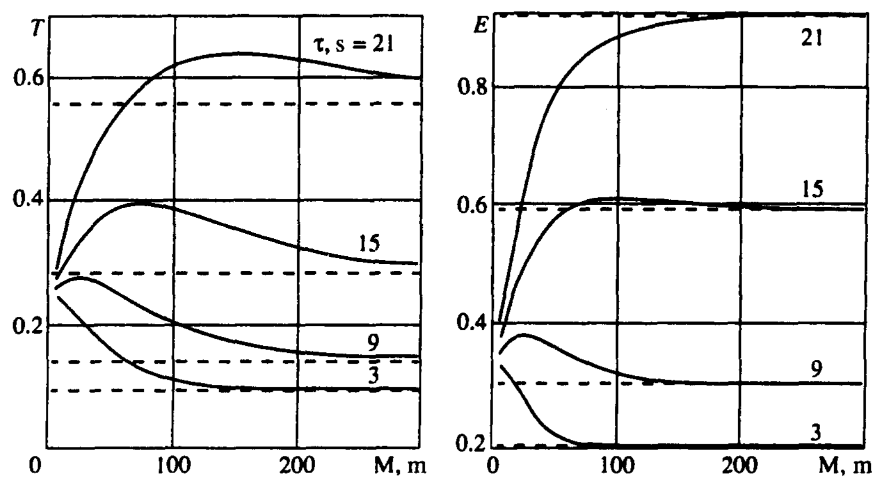

Fig. 3

hinder the motion of the water. The relations $T(\tau)$ and $E(\tau)$ are qualitatively identical. However, they differ quantitatively. The rigid support which inhibits vertical displacements of the plate, also simultaneously prevents the vertical motion of liquid particles which, under the support, are mainly displaced in a horizontal direction. This transformation of the motion leads to additional reflection of the wave. Hence, the reflection coefficient of the support is much greater than would be expected on the basis of a simple comparison of the intensity of the two energy transfer channels.

The graphs in Fig. 3 illustrate the dependence of the energy transmission coefficient and the relative flow of energy in the water on the reservoir depth for different period of the incident wave. The dashed lines correspond to the case of infinitely deep water. It can be seen that the transmission coefficient of the wave changes non-monotonically as the reservoir depth increases and that a maximum of the transmission coefficient is observed at a certain depth. The reservoir depth corresponding to this maximum increases as the period becomes longer. The relative flow of energy in the water changes in a similar way. As the reservoir depth increases, an enlargement of the "liquid" energy transfer channel initially occurs, as a result of which the transmission coefficient increases. At the same time, the effect of the bottom is reduced to amplifying the horizontal displacement of the liquid particles for which the support is not a barrier. This role of the bottom decreases in the case of reservoirs of great depth and the transmission coefficient is somewhat reduced.

The traditional model [1–8] of an ideal liquid was used in the calculations. Within the framework of this model, the friction of the liquid on its fixed boundaries and, in particular, on the reservoir bottom are neglected. Friction on the bottom, particularly when account is taken of the irregularities in its relief, impedes the horizontal motions of the layer of liquid close to the bottom. Hence, the real values of the transmission coefficient of the wave in the case of shallow reservoirs may turn out to be somewhat smaller than the calculated values.

Values of the lengths of waves of different periods for various values of the reservoir depth are given in Table 1.

Table 1

M, m	τ, s = 3	5	7	9	11	13	15	17	21
5	98.3	117.0	131.5	143.6	154.2	163.8	172.6	180.9	196.2
10	120.2	144.1	162.9	179.0	193.5	206.9	219.7	231.9	255.7
20	136.0	165.3	188.8	209.6	228.9	247.4	265.7	284.0	321.4
30	142.4	175.2	202.0	226.3	249.5	272.4	295.5	319.3	369.4
50	146.7	183.3	214.6	244.0	273.4	303.7	335.7	369.6	443.4
90	147.7	186.3	220.8	255.0	291.4	331.8	377.5	428.7	543.0
190	147.7	186.6	221.6	257.1	296.3	342.8	400.5	472.4	652.5
300	147.7	186.6	221.6	257.1	296.4	343.2	401.8	476.9	680.4

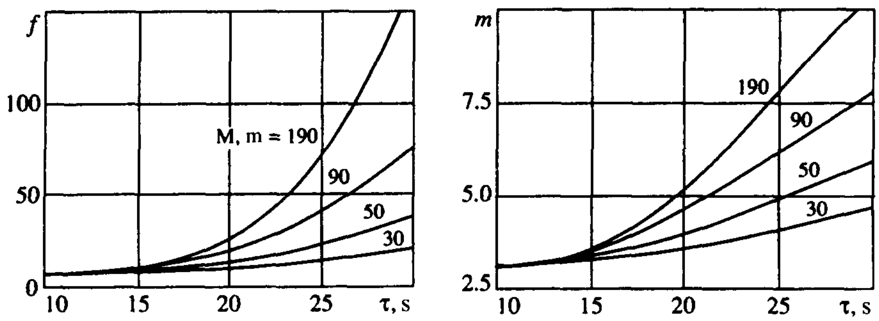

Fig. 4

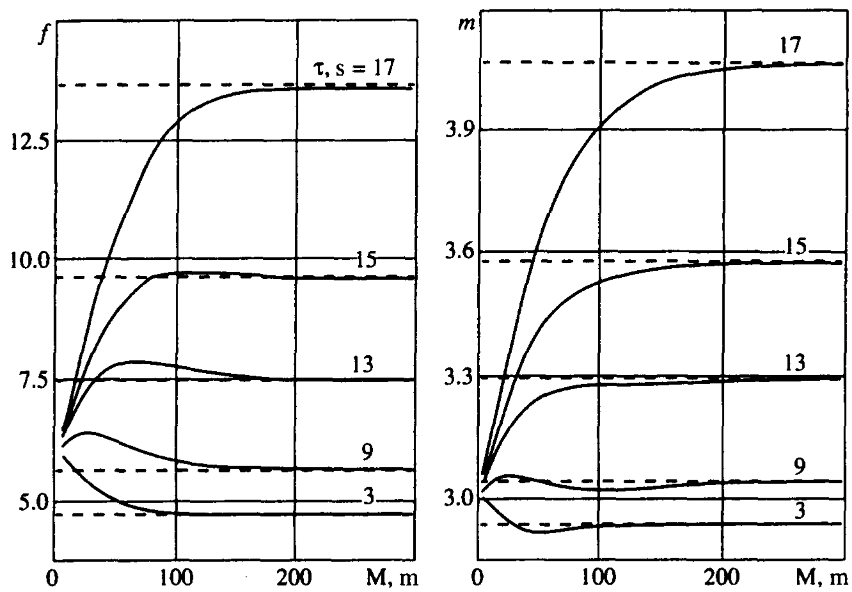

Fig. 5

5. INTERNAL FORCES IN THE SUPPORT

It is well known that the bending moment and the shearing force are characterized by discontinuities in the second and third derivatives of the function $\zeta(x)$ at zero. We use the equation for the equilibrium of the plate (1.4) and obtain

$$D\zeta'''(x) = I(x) + \frac{A}{2}\operatorname{sign} x + B\delta(x) + C_1$$

$$D\zeta''(x) = \int I(x)dx + \frac{A}{2}|x| + \frac{B}{2}\operatorname{sign} x + C_1 x + C_2$$

$$I(x) = \int (\rho h\omega^2\zeta(x) + P(x,0))dx$$

The relations for the discontinuities in the third and second derivatives of the function $\zeta(x)$ at zero

$$D[\zeta'''(x)]|_{x=0} = A, \quad D[\zeta''(x)]|_{x=0} = B$$

follow from the continuity of the functions $\zeta(x)$ and $P(x, 0)$ and the property of the concentration of the carrier $\delta(x)$ at zero. Hence, the coefficients A and B are the shearing force and the bending moment in the support, apart from the sign.

Graphs are shown in Figs 4 and 5 which reflect the effect of the period of the incident wave and the reservoir depth on the internal forces developed in the support. Here, f and m are the moduli of the normalized internal forces, that is, the shearing force and the bending moment. The normalized coefficients for the shearing force and the bending moment were calculated using the formulae

$$C_f = \frac{Di}{\omega}\frac{\partial^4\Phi_0}{\partial x^3\partial z}\bigg|_{x=0,z=0}, \quad C_m = \frac{Di}{\omega}\frac{\partial^3\Phi_0}{\partial x^2\partial z}\bigg|_{x=0,z=0}$$

Comparison of these graphs shows a similarity in the nature of the behaviour of the normalized internal forces, the energy transmission coefficient of the wave and the relative flow of energy in the water when the period of the incident wave and the reservoir depth are changed.

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